Lift-and-project methods and their application to optimal and optimization-based control

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Abstract

Lift-and-project methods draw on the concept of *lifting* a difficult problem into a higher dimensional space where some structure can be exploited (most often, a linear structure), and then *projecting* it onto a new space where numerical computations are tractable while preserving (at least partially) the interesting structure of the lifting. Such methods have proved very useful for studying and optimizing nonlinear control systems with little to no structure in the first place, either in continuous or discrete time. This session intends to provide the widest possible overview of the ongoing research using lift-and-project optimization methods to tackle difficult control and decision problems.

1 Motivation and relevance

Many real-life control systems exhibit highly nonlinear behaviors, and often lack any structure to draw upon for their study. Optimization over such systems can quickly become very difficult: costly simulations, intractable optimality conditions, controlability and stability issues... Most of the time, one is faced with hard, nonconvex optimization problems. In this context, lift-and-project methods can prove very useful, as they introduce structure into the considered problems, in order to make them easier to solve.

Lift-and-project methods originally referred to a field of combinatoric optimization that gained momentum in the early 1990s. At that time, the method was applied to integer programs (IP) in n variables that had been relaxed into linear programs (LP) on $\mathbb{R}^n$ by getting rid of integrity constraints. In order to reduce the (most often large) relaxation gap between the IP and the LP, the idea consists in using a *lift-and-project operator*: the operator would lift the feasible polytope of the LP into a higher dimension space (usually a space of matrices), introduce new, convex constraints (namely: *lift-and-project cuts*) that would approximate the integrity constraints of the IP, and then re-project the resulting feasible set onto $\mathbb{R}^n$ to obtain a new, convex feasible set, *i.e.* a convex optimization problem, with a (potentially) smaller relaxation gap than the initial LP relaxed problem. Seminal contributions in this area are [19, 15, 3, 2, 13, 4, 1] and introduced several *hierarchies* of lift-and-project operators.

In particular, the contribution of [13] actually stems from a wider idea [12] that not only applies to combinatorial optimization, but also to a number of other problems, including the optimal control problem (OCP) [14], stability analysis of control systems [6], controlled invariant sets [9], control in infinite dimension [10]. The generic idea is to write the considered problem under the form of an infinite dimensional LP on Borel measures (the *lift* part of the framework), and then to use some real algebraic geometry results [18] to derive a hierarchy of convex semidefinite programs (SDP) that can be solved numerically (*project* part), and are proved to converge towards the initial problem in several ways [20]. This extremely rich framework thus allows for addressing a variety of optimization problems that arise in control, heavily relying on Liouville's PDE and the associated transfer operator (that generates its solution).

Another famous example of lift-and-project method is the use of Koopman operators to represent a discrete or continuous time control system. The Koopman operator is actually adjoint to the aforementioned transfer operator, so that the Lasserre approach and the Koopman framework are somehow in duality. The key idea is to represent a dynamical system through the linear composition operation of its flow on functions, and to deduce properties of the system from the spectral properties of this composition operator (named Koopman operator, see [7, 5, 8]). Koopman operators have since then successfully been used for model predictive control (MPC) of nonlinear systems, see [11, 17] and the references therein.

Another promising field of lift-and-project methods is more related to learning, although it has intersections with the previously presented research topics. It consists of using reproducing kernel

Hilbert spaces (RKHS) to synthesize a control law. Briefly speaking, RKHS are used to view a nonlinear control problem in terms of approximating a function within a given (infinite dimensional) Hilbert space of functions (*lift*), as a linear combination of a finite number of kernel evaluations (*project*) that are extracted from measurement data. This rich framework ahs already been applied to combine RKHS and MPC [16] into a new framework called Kernel Predictive Control (KPC).

Of course, this list is not exhaustive, and many research topics in optimization and control could also be classified within the family of lift-and-project methods, which are characterized by the very versatile idea of lifting a problem to introduce a useful structure and then project the structured problem onto a tractable finite dimensional space. This invited session is open and we encourage any interested contributor to submit an article presenting recent advances in this field.

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